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HIERARCHICAL CLASSES MODELING OF RATING DATA

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Hierarchical classes (HICLAS) models constitute a distinct family of structural models for N-way N-mode data. All members of the family include N simultaneous and linked classifications of the elements of the N modes implied by the data; those classifications are organized in terms of hierarchical, if–then-type relations. Moreover, the models are accompanied by comprehensive, insightful graphical representations. Up to now, the hierarchical classes family has been limited to dichotomous or dichotomized data. In the present paper we propose a novel extension of the family to two-way two-mode rating data (HICLAS-R). The HICLAS-R model preserves the representation of simultaneous and linked classifications as well as of generalized if–then-type relations, and keeps being accompanied by a comprehensive graphical representation. It is shown to bear interesting relationships with classical real-valued two-way component analysis and with methods of optimal scaling.

Key words: rating data, hierarchical classes, two-mode clustering.

The family of hierarchical classes (acronym: HICLAS) models, as introduced by De Boeck and Rosenberg (1988), and further extended by Van Mechelen et al. (1995), Leenen et al. (1999), Ceulemans et al. (2003), and Ceulemans and Van Mechelen (2004), constitutes a distinct family of structural models for binary N-way N-mode data $\underline{\mathbf{D}}$. Hierarchical classes analysis of a binary $I_1 \times I_2 \times \cdots \times I_N$ data array $\underline{\mathbf{D}}$ approximates $\underline{\mathbf{D}}$ with a same-sized binary reconstructed data array $\underline{\mathbf{M}}$ that is represented by an HICLAS model. An HICLAS model implies N binary $I_n \times P_n$ matrices $\mathbf{A}_n$ $(n = 1, \ldots, N)$ and possibly one binary $P_1 \times P_2 \times \cdots \times P_N$ array $\underline{\mathbf{G}}$. The matrices $\mathbf{A}_n$ are called bundle matrices, the array $\underline{\mathbf{G}}$ is called the core and $(P_1, P_2, \ldots, P_N)$ is called the rank of the HICLAS model. Three types of relations implied by $\underline{\mathbf{M}}$ are represented by the bundle matrices and the core: equivalence, hierarchy, and association (for examples: see the next section).

Equivalence relations are defined on each of the modes of $\underline{\mathbf{M}}$ as follows: Each element of the nth mode corresponds with an $I_1 \times \cdots \times I_{n-1} \times I_{n+1} \times \cdots \times I_N$ subarray of $\underline{\mathbf{M}}$; two elements of the nth mode are equivalent iff their corresponding subarrays are identical. By the representation of these equivalence relations, and the resulting partitions into equivalence classes, an HICLAS model includes N simultaneous classifications of the elements of each of the modes of $\underline{\mathbf{M}}$.

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Similarly, *hierarchical relations* are defined on each of the modes of $\underline{\mathbf{M}}$. The latter relations are of the if–then type and constitute quasi-orders (i.e., reflexive and transitive relations) on the elements of each mode, or partial orders (i.e., reflexive, transitive and antisymmetric relations) on the corresponding equivalence classes. One element/class of the nth mode is hierarchically below a second element/class of that mode if the subarray corresponding to the first is less than or equal to the subarray corresponding to the second (in terms of the natural pointwise order defined on arrays: $\underline{\mathbf{A}} \leq \underline{\mathbf{A}}'$ iff $\forall i_1, \ldots, i_\nu : a_{i_1 \ldots i_\nu} \leq a'_{i_1 \ldots i_\nu}$). By the representation of these hierarchical relations, the N simultaneous classifications included in a hierarchical classes model turn into N hierarchically organized partitions. Although asymmetric, implicational relations are often of key substantive interest in psychological research, the formal models and methods of analysis that can properly deal with them are only few in number. The representation of the N if–then-type hierarchies may therefore be considered an important contribution of the hierarchical classes approach.

The *association relation* is the N-ary relation among the N modes of $\underline{\mathbf{M}}$ as defined by the 1-entries in $\underline{\mathbf{M}}$. Alternatively, this relation may be defined in terms of classes as an N-ary relation between the N partitions implied by $\underline{\mathbf{M}}$; the association relation may therefore be considered a linkage system between the N hierarchically organized partitions. By the representation of the association relation, $\underline{\mathbf{M}}$ may be fully reconstructed from a hierarchical classes model.

Various hierarchical classes models have been proposed, which differ in the way they represent the three types of relations as outlined above. Each of the models is accompanied by an insightful graphic representation that gives a comprehensive account of the three types of relations.

Up to now, the hierarchical classes family has been limited to binary (0/1) data. To deal with polytomous data and, in particular, with rating data with integer values in the set $\mathbb{V} = \{0, 1, \ldots, V\}$, up to now a dichotomization of the raw data was done prior to the actual hierarchical classes analysis. In order to avoid the resulting loss of information, one may consider using multiple dichotomizations, which, for example, in the case of two-way two-mode object by attribute data, come down to a replacement of each single attribute by a series of dummy variables; this strategy, however, as a side-effect, results in an expansion of one of the modes of the data, which may typically hamper the transparency and interpretability of the corresponding hierarchical classes models.

In the present paper we propose a novel extension of the hierarchical classes approach to two-way two-mode rating data (acronym: HICLAS-R). The extended hierarchical classes model includes the representation of natural generalizations of the relations of equivalence, hierarchy, and association; moreover, it is accompanied by a comprehensive graphical representation. The extended model can further be shown to bear interesting links with the two-way real-valued principal component analysis model as well as with methods of optimal scaling.

The remainder of this paper is organized as follows: Section 1 briefly recapitulates the original disjunctive hierarchical classes model for two-way two-mode binary data as originally introduced by De Boeck and Rosenberg (1988). Section 2 introduces the novel hierarchical classes model for two-way two-mode rating data, and Section 3 discusses the associated data analysis. The HICLAS-R model is illustrated with an empirical application in Section 4. Section 5 discusses the relation of the novel hierarchical classes model with other models both inside and outside the hierarchical classes family, as well as possible model extensions.

1. The Disjunctive Hierarchical Classes Model for Two-Mode Binary Data

A rank P disjunctive two-way HICLAS model for an $I \times J$ reconstructed data matrix $\mathbf{M}$ implies a binary $I \times P$ bundle matrix $\mathbf{A}$ and a binary $J \times P$ bundle matrix $\mathbf{B}$. The relations of equivalence, hierarchy, and association are represented by these bundle matrices as follows: (a) Equivalent elements of Mode 1 (2) have identical rows in **A** (**B**). (b) Two elements/classes of Mode 1 (2) are hierarchically related iff for their row vectors in **A** (**B**) it holds that the vector of the first is less than or equal to the vector of the second (in terms of a pointwise order). (c) For the association of the ith element of Mode 1 with the jth element of Mode 2, it holds that

$$m_{ij} = \bigoplus_{p=1}^{P} a_{ip} b_{jp}, \tag{1}$$

where $\bigoplus$ denotes a Boolean sum. Note that from (1) one may immediately derive the association relation between the classes of the two modes.

At this point, three remarks can be made. First, (1) can be trivially rewritten as

$$m_{ij} = \operatorname*{Max}_{p=1}^{P} a_{ip} b_{jp}, \tag{2}$$

where $\text{Max}_{p=1}^{P}$ denotes the maximum of the subsequent expression over all values of p ranging from 1 to P. Second, whereas (1) and (2) do not explicitly include a core array, they can be easily rewritten as

$$m_{ij} = \operatorname*{Max}_{p=1}^{P} \operatorname*{Max}_{q=1}^{Q} a_{ip} b_{jq} g_{pq} \tag{3}$$

with the core matrix **G** being an identity matrix. Note that an identity core implies a one-to-one relationship between the Mode 1 and Mode 2 bundles. Third, (1) and (2) imply that

$$m_{ij} = 1 \quad \text{iff} \quad \exists p : (a_{ip} = b_{jp} = 1), \tag{4}$$

meaning that elements of Mode 1 and Mode 2 are associated iff they belong to *at least* one pair of corresponding bundles. In this, "at least," that is, the existential quantifier (or, alternatively, the Max-operator in (2) and (3)), formalizes the disjunctive nature of the model. Note that a dual conjunctive model may be constructed making use of the universal quantifier (or, alternatively, the Min-operator) (for details, see Van Mechelen et al., 1995).

The disjunctive HICLAS model can be given a comprehensive graphic representation. More in particular, Hasse diagrams can be derived from the quasi-orders implied by **A** (**B**); such diagrams are directed graphs, the vertices of which are equivalence classes and the edges of which link each class to its immediate successor in the order (the remaining order links being obtained by making use of transitivity). In the Hasse diagram of each hierarchy, all bottom classes (i.e., all classes that belong to a single bundle only) are further included, irrespective of whether they are empty or not, in view of the representation of the association relation. As to this relation, a dashed link is drawn between each pair of corresponding bundles/bottom classes of the Mode 1 and Mode 2 hierarchies. (Note that, whereas in some cases empty bottom classes can be omitted, this is in general not allowed to preserve a correct representation of the association relation.)

To illustrate, we will make use of the leftmost matrix in Table 1 as a hypothetical reconstructed data matrix **M**. Table 2 contains a rank 2 disjunctive HICLAS model for **M**. A graphic representation of this model is presented in Figure 1. The upper Hasse diagram in the figure represents the hierarchically ordered partition of the first mode, and the lower Hasse diagram that of the second mode; note that the Hasse diagram of the second mode is represented upside down. The association relation can be read from the figure as follows: An element/class of the first mode is associated with an element/class of the second iff it is linked to it via a downward path of links and a dashed line.

TABLE 1.
Hypothetical reconstructed binary data matrix.

	Mode 2			
Mode 1	α	β	γ	δ
a	1	1	1	1
b	1	1	1	1
c	1	0	0	1
d	0	1	1	1
e	0	0	0	0
f	0	0	0	0

TABLE 2.
Disjunctive rank 2 hierarchical classes model for reconstructed data matrix of Table 1.

	A			**B**	
a	1	1	α	1	0
b	1	1	β	0	1
c	1	0	γ	0	1
d	0	1	δ	1	1
e	0	0			
f	0	0			

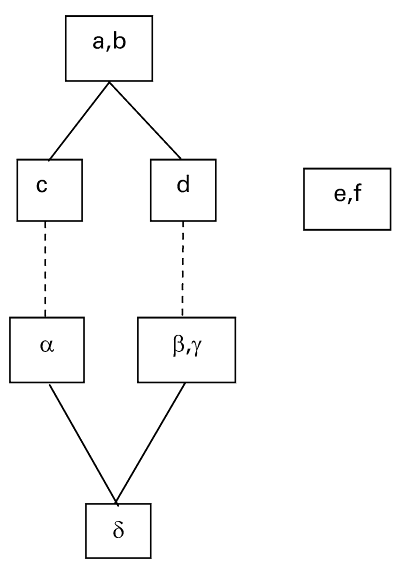

FIGURE 1.
Graphic representation of the disjunctive two-way HICLAS model of Table 2. Equivalent elements are displayed in boxes. The hierarchy of the row classes is represented directly in the upper half of the figure, whereas the hierarchy of the column classes is represented upside down in the lower half. A row and column element are associated if they are linked by at least one downward path of links.

2. A Disjunctive HICLAS Model for Two-Mode Rating Data

We assume a two-way two-mode $I \times J$ data matrix $\mathbf{D}$ with integer values ranging from 0 to V. Hierarchical classes analysis will approximate $\mathbf{D}$ by an $I \times J$ reconstructed data matrix $\mathbf{M}$ with integer values ranging from 0 to V that can be represented by an HICLAS-R model. As a guiding example for the remainder of this section, we consider the hypothetical reconstructed rating data matrix that is presented in Table 3.

We want the HICLAS-R model to represent three aspects of $\mathbf{M}$ that constitute straightforward generalizations of the three relations represented by HICLAS models for binary data: (a) equivalence, (b) hierarchy, and (c) association. In particular: (a) Two Mode 1 (2) elements are *equivalent* iff they correspond with identical rows (columns) in $\mathbf{M}$. (b) One element/class of Mode 1 (2) is *hierarchically below* another element/class of that mode iff the vector in $\mathbf{M}$ defined by the first is less than or equal to the vector defined by the second (in terms of a pointwise order); this means that, *if* the first element/class is associated with an element/class e of the other mode at some level of association strength, *then* the second element/class is associated with e at a level of association strength that is at least as high. (c) *Association* refers to the mapping defined by the entries in $\mathbf{M}$, which links each pair of a row i and a column j to some value m_{ij} in the value set $\mathbb{V} = \{0, 1, \ldots, V\}$. For example, from Table 3, it appears that b and c are equivalent, that c is hierarchically below d, and that the pair (c, β) is mapped onto 2.

A rank (P, Q, R) disjunctive HICLAS-R model for an $I \times J$ reconstructed data matrix $\mathbf{M}$ implies a binary $I \times P$ bundle matrix $\mathbf{A}$, a binary $J \times Q$ bundle matrix $\mathbf{B}$, and a rating-valued $P \times Q$ core matrix $\mathbf{G}$ that takes exactly $R \leq V$ different nonzero values from the value set $\mathbb{V} = \{0, 1, \ldots, V\}$. The relations of equivalence, hierarchy, and association are represented by the bundle matrices as follows: (a) Equivalent elements of Mode 1 (2) have identical rows in $\mathbf{A}$ ($\mathbf{B}$). (b) Two elements/classes of Mode 1 (2) are hierarchically related iff for their row vectors in $\mathbf{A}$ ($\mathbf{B}$) it holds that the vector of the first is less than or equal to the vector of the second (in terms of a pointwise order). (c) For the association of the ith element of Mode 1 with the jth element of Mode 2, it holds that

$$m_{ij} = \operatorname*{Max}_{p=1}^{P} \operatorname*{Max}_{q=1}^{Q} a_{ip} b_{jq} g_{pq}. \tag{5}$$

Equation (5) implies that a Mode 1 element is associated with a Mode 2 element at the maximum level of association indicated in the core matrix G for a pair of bundles to which the two elements belong.

Table 4 contains a disjunctive rank (3, 3, 3) HICLAS-R model for the reconstructed rating data matrix of Table 3. More in particular, the table contains the bundle matrices and the rating-valued core matrix $\mathbf{G}$.

The HICLAS-R model can be given a comprehensive graphic representation: Hasse diagrams can again be derived from the quasi-orders implied by $\mathbf{A}$ ($\mathbf{B}$). In the Hasse diagram of each hierarchy, all bottom classes (i.e., all classes that belong to a single bundle only) are again

TABLE 3.
Hypothetical reconstructed rating data matrix.

	Mode 2		
Mode 1	α	β	γ
a	1	0	1
b	0	2	1
c	0	2	1
d	0	3	1

TABLE 4.
Disjunctive rank (3, 3, 3) HICLAS-R model for reconstructed rating data matrix of Table 3.

	A				**B**			**G**		
a	1	0	0	α	1	0	0	1	0	0
b	0	1	0	β	0	0	1	0	1	2
c	0	1	0	γ	1	1	0	0	0	3
d	0	1	1							

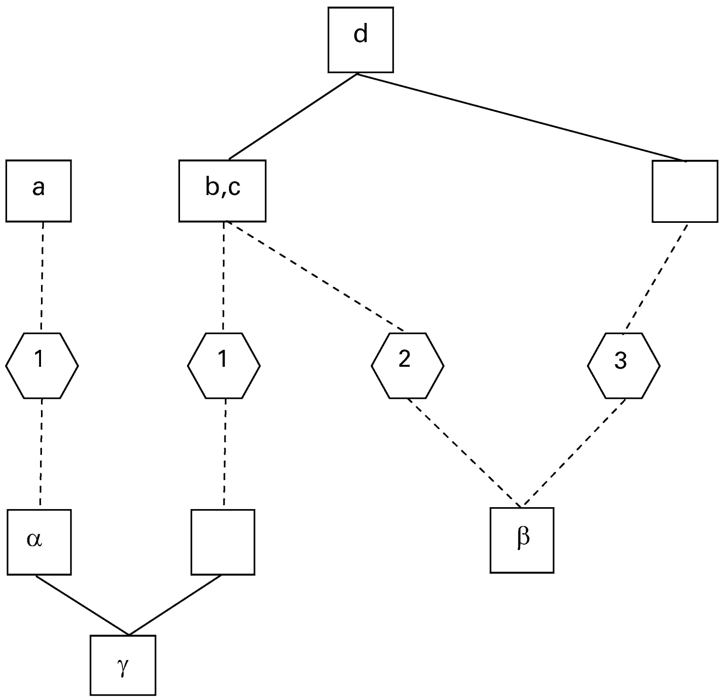

FIGURE 2.
Graphic representation of the HICLAS-R model of Table 4. Equivalent elements are displayed in boxes. The hierarchy of the row classes is represented directly in the upper half of the figure, whereas the hierarchy of the column classes is represented upside down in the lower half. A row and column element are associated at the level of the maximum of all diamond values on downward paths linking them.

further included, irrespective of whether they are empty or not, in view of the representation of the association relation. As to this relation, a dashed link is drawn between each pair of bundles/bottom classes of the Mode 1 and Mode 2 hierarchies corresponding with a nonzero value in **G**; the latter value is further entered as a label in the diamond on the link. As an example, Figure 2 contains a graphic representation of the HICLAS-R model of Table 4. From the graphic representation, one may immediately read, in the usual way, the relations of equivalence and hierarchy in the two modes of the reconstructed rating data matrix. Regarding association, it holds that two elements are associated at the level of the maximum of the values in the diamonds on all downward paths linking them with one another.

3. HICLAS-R Data Analysis

3.1. Loss Function

The aim of a disjunctive HICLAS-R analysis in rank (P, Q, R) of a rating-valued data matrix $\mathbf{D}$ is to approximate $\mathbf{D}$ as closely as possible with a reconstructed rating-valued data matrix $\mathbf{M}$ that can be represented by a rank (P, Q, R) HICLAS-R model. Closeness can be formalized in terms of a Minkowski norm-based loss function. A natural candidate within the discrete framework of the HICLAS-R model is the L_1 norm, yielding the least absolute deviations loss function L,

$$L = \sum_{i,j} |m_{ij} - d_{ij}|. \tag{6}$$

Note that this loss function implies that the rating-valued data are dealt with as if they were measured on an absolute scale.

3.2. Equivalent Model Formulation

To optimally fit a rank (P, Q, R) HICLAS-R model to a data set at hand, it is most useful to rely on an equivalent formulation of the HICLAS-R model that may be obtained by means of the recoding function t that maps the set of all $I \times J$ matrices with entries in $\mathbb{V} = \{0, 1, \ldots, V\}$ into the set of all binary $I \times J \times V$ arrays:

$$\begin{aligned} &\mathbb{V}^{I\times J} \to \{0,1\}^{I\times J\times V}, \\ &\mathbf{M} \mapsto t(\mathbf{M}), \quad \text{with} \quad t(\mathbf{M})_{ijv} = \begin{cases} 1 & \text{if } m_{ij} \geq v, \\ 0 & \text{otherwise,} \end{cases} \quad v \in \{1, \ldots, V\}. \end{aligned} \tag{7}$$

The transformation t can be considered a standard dummy recoding according to an ordinal coding scheme, with, however, the arrangement of the recoded data into a three-way array as a special feature.

Note that (7) implies that

$$m_{ij} = \sum_{v=1}^{V} t(\mathbf{M})_{ijv}. \tag{8}$$

Note also that $t(\mathbf{M})$ satisfies the following reversing order property:

$$\forall v' \leq v : t(\mathbf{M})_{ijv} \leq t(\mathbf{M})_{ijv'}. \tag{9}$$

Furthermore, the transformation t preserves the relations of equivalence in the two data modes, in that two elements of Mode 1 (2) are equivalent in $\mathbf{M}$ iff the corresponding elements of Mode 1 (2) in $t(\mathbf{M})$ are equivalent; similarly, t also preserves the hierarchical relations in that one element of Mode 1 (2) is hierarchically below a second element of that mode iff the corresponding elements in $t(\mathbf{M})$ are hierarchically related.

A rank (P, Q, R) HICLAS-R model for an $I \times J$ reconstructed rating data matrix $\mathbf{M}$ is now equivalent with a rank (P, Q, R) disjunctive Tucker3-HICLAS model of the recoded binary $I \times J \times V$ array $t(\mathbf{M})$ (Ceulemans et al., 2003). Such a Tucker3-HICLAS model implies binary $I \times P$, $J \times Q$, and $V \times R$ bundle matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, and a binary $P \times Q \times R$ core array $\underline{\tilde{\mathbf{G}}}$. The relations of equivalence and hierarchy in $t(\mathbf{M})$ are represented by the bundle matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Regarding association, it holds that

$$t(\mathbf{M})_{ijv} = \operatorname*{Max}_{p=1}^{P} \operatorname*{Max}_{q=1}^{Q} \operatorname*{Max}_{r=1}^{R} a_{ip} b_{jq} c_{vr} \tilde{g}_{pqr}. \tag{10}$$

Equation (10) can be shown to be equivalent with the HICLAS-R association rule (5), with

$$g_{pq} = \operatorname*{Max}_{r=1}^{R}\left(\tilde{g}_{pqr}\sum_{v=1}^{V} c_{vr}\right). \tag{11}$$

3.3. Algorithmic Strategy

In this subsection we describe an algorithmic strategy to fit a rank (P, Q, R) HICLAS-R model to a data set $\mathbf{D}$ at hand making use of loss function (6) and of the equivalent model formulation as outlined in Section 3.2 (see also Ceulemans and Van Mechelen, 2003). The architecture of the algorithm is as follows:

(i) Step 1: Binary recoding of data by means of transformation (7).
(ii) Step 2: Fitting of a rank (P, Q, R) Tucker3 HICLAS model to transformed data $t(\mathbf{D})$, subject to the constraint that the reconstructed $t(\mathbf{D})$ satisfies (9).
(iii) Step 3: Converting the final result of Step 2 into a regular HICLAS-R model formulation making use of (11).

The key step of the algorithm, Step 2, further consists of the following substeps:

(i) Step 2a: Initial configurations for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are obtained through INDCLAS analyses on $t(\mathbf{D})$ (Leenen et al., 1999) and through disjunctive HICLAS analyses on the three possible matricized versions of $t(\mathbf{D})$.
(ii) Step 2b: $\underline{\tilde{\mathbf{G}}}$, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are (re-)estimated making use of an alternating conditional estimation procedure.
(iii) Step 2c: The final $\mathbf{A}$ and $\mathbf{B}$ matrices that result from Step 2c are transformed so as to make them represent correctly the equivalence and hierarchical relations in the reconstructed $t(\mathbf{D})$.

The conditional (re-)estimations of $\underline{\tilde{\mathbf{G}}}$, $\mathbf{A}$, $\mathbf{B}$ in Step 2b are achieved through Boolean regressions (Leenen and Van Mechelen, 2001); for $\mathbf{A}$ and $\mathbf{B}$ these regressions can be done for each row separately due to a separability property of loss function (6). The reestimation of $\mathbf{C}$ is to be done taking into account constraint (9). This is achieved by subsequently looking for the optimal values for each of the rows of $\mathbf{C}$ (by means of an enumerative search) under the restriction that $\forall r = 1, \ldots, R : c_{vr} \le c_{(v-1)r}$.

3.4. Performance of the HICLAS-R Algorithm

The performance of the HICLAS-R algorithm was evaluated in an extensive simulation study based on simulated data sets that consisted of true structures plus error (Ceulemans and Van Mechelen, 2003). From the latter, it appeared that, on average across 9660 simulated data sets, the algorithm succeeds in finding a model that is at least as close to the data as the truth is, with the performance being better for data sets that are prone to less error and that are larger in size. With regard to recovery of the truth underlying the error-perturbed data, on average, 82.8% of the true equivalence and hierarchical relations were recovered, whereas for the association relation, the percentage of correctly recovered entries of the truth (transformed by means of (7)) amounts to 93.1%. For higher error levels (25% and 30%) goodness of recovery appears to be somewhat less good (i.e., 71.2% and 64.6% for the equivalence plus hierarchical relations; 87.4% and 78.1% for the association relation).

4. Illustrative Application

We fitted the HICLAS-R model to data from a study on helping behavior. A group of 102 students was presented with an experimental list with 16 descriptions of everyday emergency situations with a victim that could possibly be helped by the subject. The list was constructed on the basis of a facet-theoretic design, the facets being: extent of the victim's distress (low vs. high) and the subject's expectation to get something in return for possible help (no vs. yes). This is an (abbreviated) example of a situation description: "In a very crowded grocery store you see a little boy, weeping and crying for his mum." (high distress/nothing in return). The students were asked to rate each situation with respect to the extent they would be willing to help the victim in it. For this purpose they had to use a rating scale from 0 (definitely not) through 6 (definitely yes).

The resulting 102 by 16 rating data matrix was subjected to HICLAS-R analyses in ranks $(1, 1, 1)$ through $(6, 6, 6)$. A generalized scree test on the resulting proportions of discrepancies (Ceulemans et al., 2003) suggested that either a $(2, 2, 2)$ or a $(2, 3, 3)$ solution was to be preferred. On the basis of interpretational considerations, we finally retained the $(2, 3, 3)$ solution. The latter had 13.9% discrepancies and a Jaccard goodness-of-fit value (Jaccard, 1908) of .83.

Figure 3 contains a graphic representation of the $(2, 3, 3)$ model. Regarding the situation hierarchy, it immediately appears from the figure that the situations constitute a three-level Guttman scale, which means that they imply a quantitative dimension (Gati and Tversky, 1982). In order to derive a substantive psychological interpretation for this dimension, the position on it (quantified as 1, 2, 3) was correlated with external ratings of the situations as obtained from expert judges (measured on 4-point Likert scales and averaged across judges). The two highest correlations were obtained with the expert ratings of the extent to which the situation was frustrating ($r = -.74$) and of the extent to which it was emotionally threatening for the potential helper ($r = -.73$). These correlations are remarkably high, especially given the fact that the situation Guttman scale comprised three different levels only. Apparently, overall extent of willingness to help in an emergency situation, unlike what one might expect, does not primarily depend on straightforward situation characteristics such as extent of the victim's distress (for which $r = .29$ only). Rather, willingness to help appears to be especially low in emergency situations that are frustrating or unpleasant for the potential helper.

Regarding the value set $\mathbb{V}$, in line with the model rank $(2, 3, 3)$, the HICLAS-R model contains, in addition to zero, three values only from the original seven-point rating scale (0–6). These values are: 3 (the scale midpoint), 4 (the value just above the scale midpoint), and 6 (the maximum value). We may conclude that our analysis sheds light on how the rating scale was used by our subjects, the major distinctions being: refusal to help (0), doubt (3), weakly positive answer (4), and clear willingness to help (6).

Regarding the person hierarchy, we observe that three person types can be distinguished. Unlike the situation classes, they do not constitute a Guttman scale. The characteristic response profiles of each of the person types may be read from the graphic representation in Figure 3. The profiles can also be given an alternative graphic representation, as shown in Figure 4; the construction and interpretation of this alternative representation is facilitated by the quantitative dimension underlying the situation hierarchy (although a similar representation could also be considered if the situation classes were not totally ordered). One may first note that all response profiles in Figure 4 are nondecreasing; the latter necessarily follows from the Guttman scale structure of the situation classes. Furthermore, the response profile of Person Type I reflects a rather clear-cut, categorical nature; the persons of this type are willing to admit clearly that they do not intend to help in highly unpleasant situations, whereas, at the same time, they are also willing to express a definite intention to help in lowly unpleasant situations; moreover, they do not leave room for doubt (3: midpoint score) in their response profile. Persons of Person Type II display a large amount of doubt and avoid extreme responses of any kind; as a result, they do not differentiate considerably between situations at distinct levels of unpleasantness. Finally, persons of Person Type III do not want to give clearly negative answers, whereas they do express a definite intention to help in lowly unpleasant situations; otherwise, in all situations their amount

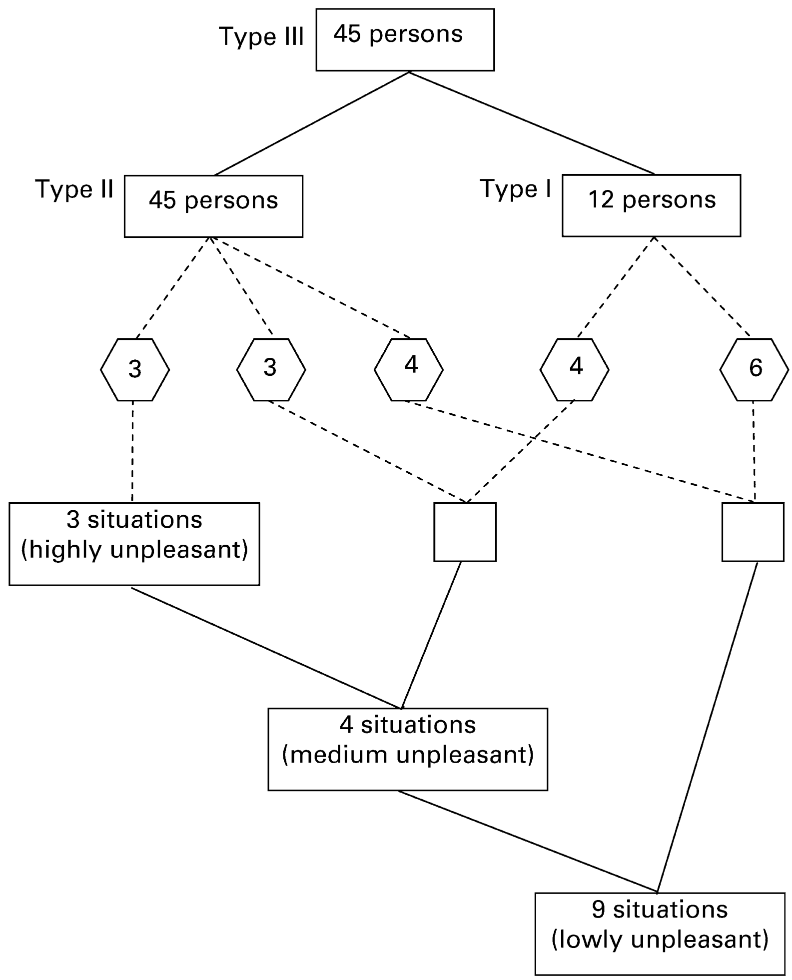

FIGURE 3.
Graphic representation of the rank (2, 3, 3) HICLAS-R model of helping data.

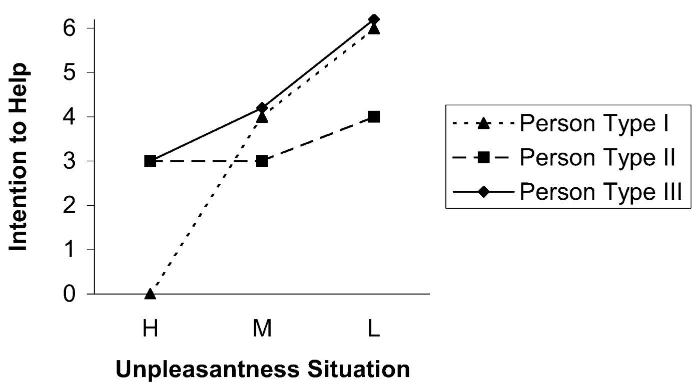

FIGURE 4.
Alternative graphic representation of helping behavior profiles of three person types from the rank (2, 3, 3) HICLAS-R model of helping data.

of willingness to help is equal to or exceeds the corresponding amount for their colleagues of Person Types I and II, which reflects the hierarchical relation between Person Type III on the one hand and Person Types I and II on the other hand.

5. Discussion

In this section we first discuss the relations of the novel HICLAS-R model with other models, both within and outside the hierarchical classes family (Section 5.1). Next we discuss possible model extensions (Section 5.2).

5.1. Relation with Other Models

5.1.1. Relation with Other HICLAS Models. In line with the other HICLAS models, the novel HICLAS-R model preserves the representation of simultaneous and linked classifications as well as of generalized if–then-type relations. Moreover, the HICLAS-R model, just like the other HICLAS models, is accompanied by a comprehensive graphical representation.

From the argument above, it immediately follows that the HICLAS-R model naturally extends the disjunctive two-way HICLAS model for binary data as developed by De Boeck and Rosenberg (1988), the latter model being obtained by putting P equal to Q and by putting the core equal to an identity matrix. Those restrictions do not imply an effective constraint in the binary case; otherwise, it is easy to see that the disjunctive HICLAS model for two-mode binary data can already be obtained by constraining the core of the HICLAS-R model to be binary.

In spite of the striking similarities between the HICLAS and HICLAS-R models, one should take care not to erroneously transfer features of the original two-mode HICLAS model to its HICLAS-R counterpart. For example, if in a regular two-mode HICLAS model the hierarchy of one mode constitutes a Guttman scale, then the same necessarily also holds for the other mode; this, however, is not the case for the HICLAS-R model, as illustrated by the model of the helping data of Section 4. Moreover, whereas in the regular two-mode HICLAS model the hierarchies of both modes necessarily have the same complexity (i.e., the same number of bottom classes or underlying bundles), and whereas that model implies a one-to-one relationship between the bundles/bottom classes of both modes, this does not have to be the case in the HICLAS-R model. As a matter of fact, the HICLAS-R model may be considered to inherit both features of the regular two-mode HICLAS model (viz., the inclusion of two bundle matrices) and of the Tucker3-HICLAS model (viz., the possibility of different numbers of bundles for different modes, and the inclusion of a core).

One may note that the HICLAS-R model can be constrained such as to reinstall a one-to-one relationship between the Mode 1 and Mode 2 bundles. The latter can be achieved by constraining the core matrix $\mathbf{G}$ in (5) to a diagonal matrix. A sufficient (though not necessary) condition for this is to assume that the three-way core array $\underline{\tilde{\mathbf{G}}}$ in (10) is a so-called “superidentity” array, which means that $t(\mathbf{M})$ is represented by an INDCLAS model (i.e., the CANDECOMP/PARAFAC counterpart of the Tucker3-HICLAS model: see Leenen et al., 1999).

5.1.2. Relation with Classical Real-Valued Principal Component Analysis. The regular disjunctive two-mode HICLAS model bears a natural relationship to the model of real-valued principal component analysis. Indeed, apart from the distinction between Boolean and non-Boolean sums, and apart from the fact that the bundle matrices of the HICLAS model are constrained to be binary, association rule (1) is identical to the model of principal component analysis. For the HICLAS-R model, at first sight, the link with principal component analysis may seem less obvious, especially given the presence of a core matrix in association rule (5). Interestingly, however, the model of real-valued two-way component analysis may be reformulated such as to include a core matrix as well (Levin, 1965). The model then reads as follows:

$$m_{ij} = \sum_{p=1}^{P} \sum_{q=1}^{P} a_{ip} b_{jq} g_{pq}, \tag{12}$$

with **A** and **B** denoting principal-axes loading matrices, both rotated to simple structure, and **G** denoting the counterrotated diagonal matrix with reciprocals of the first P singular values. Equation (12) comes very close to association rule (5), especially if one takes into account that the Max-operators in (5) can be considered generalized (Boolean) sum-operators. Apart from the constraint implied by the HICLAS-R model for the matrices **A** and **B** in (5) to be binary, the only remaining key difference between association rule (5) and the principal component model is that, in Levin's PCA, the numbers of Mode 1 and Mode 2 components are necessarily equal (which further implies that the core is a square matrix). Indeed, if the two numbers of components in (12) would be unequal, then it is easy to show that the model equation can be rewritten with identical numbers of components; this, however, is not the case for association rule (5).

In practice, in spite of the simple structure rotations implied by model (12), the **A** and **B** loadings keep taking values on a continuous scale, which may somewhat hamper the substantive interpretation of the reformulated PCA model as compared to its HICLAS-R counterpart. For example, we subjected the helping data of Section 4 to a reformulated PCA, making use of a singular value decomposition (truncated at three components) followed by a double VARIMAX rotation. In the resulting situation component matrix, sizable loadings on the first component only occurred for all but one of the lowly unpleasant situations, sizable loadings for the second component showed up for two medium unpleasant situations, and sizable loadings for the third component for two highly unpleasant situations (in addition to one medium unpleasant situation). Except for one entry, the core also reflected the general Guttman pattern as revealed by the HICLAS-R analysis. Yet, in general, because of the diffuse pattern of the person and situation loadings and because of the additive nature of the model, the adjusted principal component model appeared to be harder to interpret in this case than its HICLAS-R counterpart.

5.1.3. Relation with Methods of Optimal Scaling. A reconstructed rating data matrix of a rank (P, Q, R) HICLAS-R model contains exactly R different nonzero rating values. (Note that from the general Tucker3-HICLAS theory it follows that $R \leq P \times Q$; Ceulemans et al., 2003.) In practice, the number R will often be smaller than V (i.e., the number of nonzero values in the value set $\mathbb{V}$). Hence, the HICLAS-R model may be considered to imply a reduction of the value set to a more coarse subset of values. The latter reduction may highlight the most important distinctions on the rating scale that is being used. As such, this may reveal useful information on, for example, how the rating scale has been dealt with psychologically by the subjects under study. This is nicely illustrated by the analysis of the helping data.

One may observe a striking parallel between the latter type of substantive inferences implied by the HICLAS-R modeling and similar inferences derived from methods of optimal scaling (Gifi, 1990; Van de Geer, 1993). For example, in applications of optimal scaling one may arrive at the conclusion that a number of categories of a categorical variable under study more or less merge, whereas clear discriminations between other categories show up (e.g., Van de Geer, 1993, Vol. 2, p. 85). Interestingly, however, the HICLAS-R and optimal scaling strategies seem to arrive at those similar inferences, starting from almost opposite strategies to deal with categorical data: In optimal scaling, inferences on the most important scale distinctions and on scale use are typically based on real-valued (and, hence, more refined) quantifications of the original categorical variables at hand, whereas in HICLAS-R the same type of inferences are based on a *reduction* of the value set of the categorical variable under study. One may observe that the latter reduction strategy links up in a more straightforward way with the type of final conclusions on psychologically important scale distinctions that are derived. As an aside, one may further note that for the optimally refined quantification implied by optimal scaling methods, and for the optimally coarsened reduction of the value set implied by HICLAS-R, typically other criteria of optimality are considered: For optimal scaling methods this typically is some standard loss function stemming from the context of linear multivariate data analysis, whereas for HICLAS-R this is a least absolute deviations loss function stemming from a generalized Boolean algebra context.

5.2. Possible Extensions of the HICLAS-R Model

Various possible extensions of the HICLAS-R model could be considered, both from the point of view of the data and from that of the model.

Regarding the *data*, in the present paper HICLAS-R has been advanced as a model for two-way rating-valued data with integer values ranging from zero to some maximum value V. The proposed model, however, could be extended to a hybrid discrete-continuous model for *positive real-valued* data (such as, e.g., reaction time data). Such an extension, however, would require the search for a new mathematical framework (that should be continuous, unlike the discrete framework of (extended) Boolean matrix algebra as used up till now within the HICLAS family), as well as the development of novel types of algorithmic approaches. As a second possible data-related model extension, one may wish to represent integer-valued data with a minimum data value υ different from zero. The most straightforward HICLAS-R model extension one may consider to capture this type of data is to extend model equation (5) with an offset term υ:

$$m_{ij} = \upsilon + \operatorname*{Max}_{p=1}^{P} \operatorname*{Max}_{q=1}^{Q} a_{ip} b_{jq} g_{pq}. \tag{13}$$

From a *modeling* viewpoint, similar to the case of the regular two-way HICLAS model for binary data (Van Mechelen et al., 1995), one may formulate a dual conjunctive variant of the disjunctive HICLAS-R model as described in the present paper. Such a conjunctive variant would rely on the Min- rather than on the Max-operator. From a conceptual viewpoint, a conjunctive model may be preferable in cases where one conjectures a mechanism involving a minimum-rule to underlie the data (as in a horse-race-like scenario, in which only the—minimal—time of the winning horse is recorded); from a pragmatic viewpoint, depending on the data set at hand and on a prespecified rank, a disjunctive or a conjunctive model may yield a better fit to the data.

Finally, one might wish to consider various types of constrained HICLAS-R models, for instance, in a confirmatory approach to test a priori hypotheses stemming from substantive theories or from previous empirical research (for an extensive discussion of this topic, see Ceulemans et al., 2004). As an example, rather than deriving "by accident" a Guttman scale structure for one of the data modes (as was the case in the analysis of the helping data in Section 4), one may wish to impose such a structure in an a priori way. Otherwise, as a side effect, a Guttman scale constraint may also facilitate the construction of alternative graphic representations of the resulting HICLAS-R models as shown in Figure 4.

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